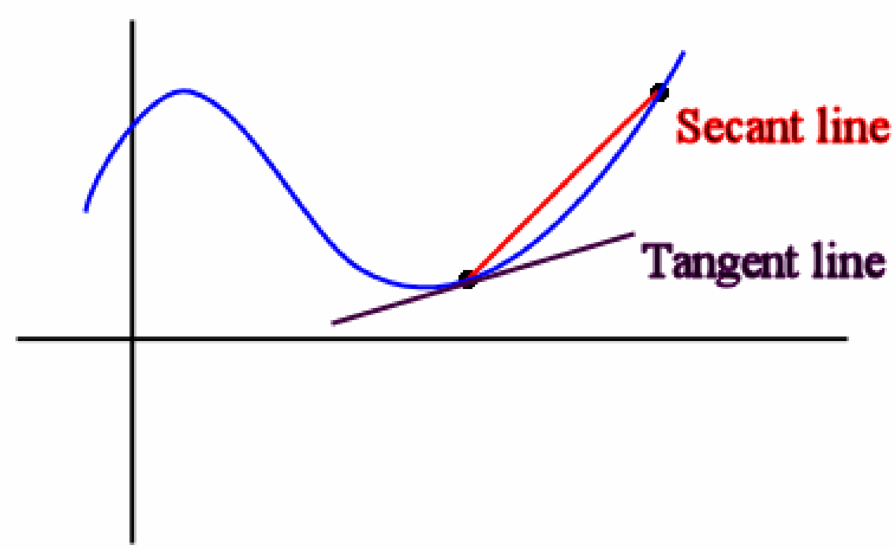
Section 1.5 Definition of Derivatives (Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

A **secant line** is a line that connects two points on the graph of a function.

A **tangent line** is a line that touches the graph of a function at a point, matching the curve's slope at the point.

Here is an image that shows a tangent and a secant line:



This is very important!!!

These are all equivalent:

- Slope of a tangent line at $x = a$

- The value of a derivative at $x = a$

- The instantaneous rate of change at $x = a$

In section 1.4 we learned a formula to find the derivative (instantaneous rate of change) of a function $f(x)$ for a specific value of $x = a$. Here is the formula from section 1.4.

$$\textbf{\textit{Instantaneous rate of change (derivative)}} = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$$

If we replace the number "a" with the variable "x" we get the formula for the derivative of a function for any value of "x".

$$\textbf{\textit{Derivative formula}} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\textbf{\textit{Instantaneous rate of change formula}} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\textbf{\textit{Formula to find slope of a tangent line}} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example - Tangent line / derivative:

Below is a graph of the function $f(x)$ and the graph of the tangent line to $f(x)$ at the point $(3,0)$

a) Find the slope of the tangent line at $(3,0)$

Pick any two points on the tangent line and apply the slope formula.

$points\ \ (3,0)\ and\ (5,6)$

$slope\ of\ tangent\ line = \frac{6-0}{5-3} = \frac{6}{2} = 3$

Answer: The slope of the tangent line is $\boldsymbol{m = 3}$.

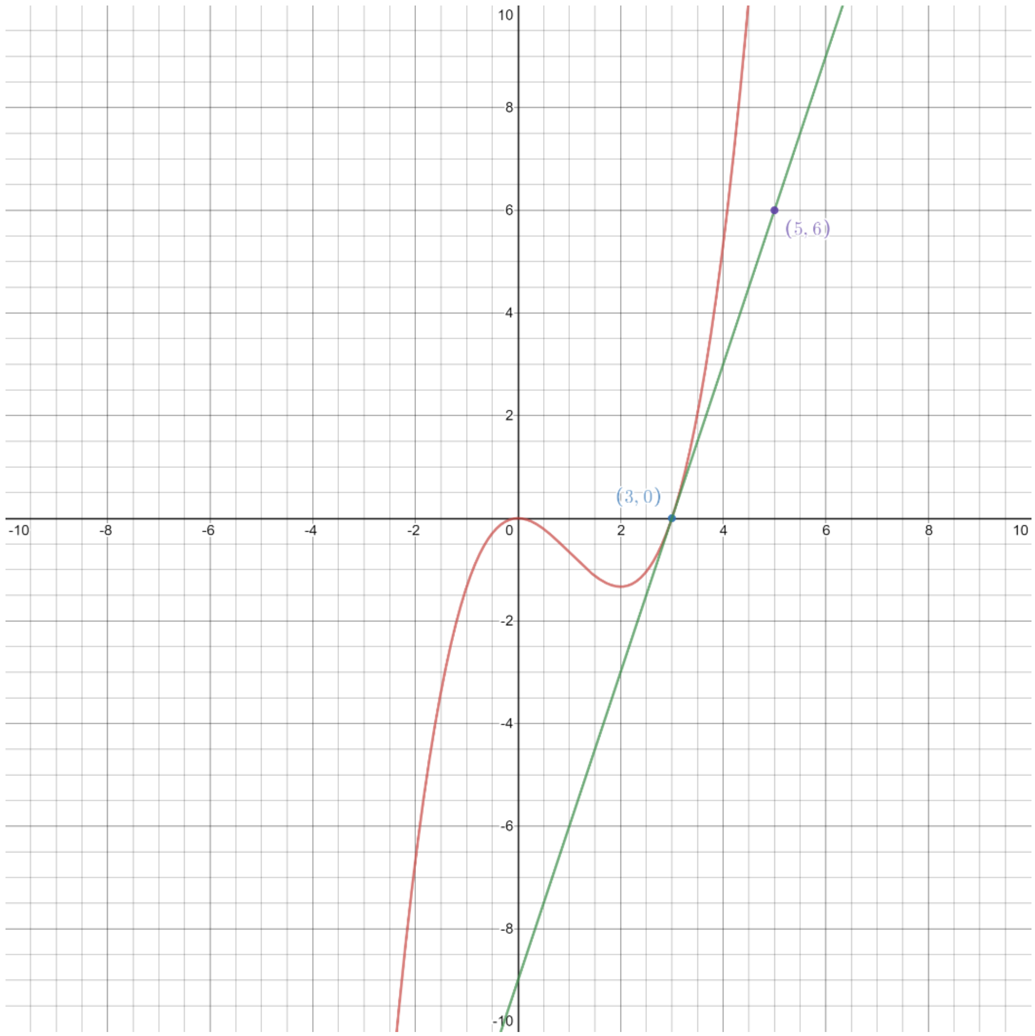
b) Find the $f'(3)$
(that is find the value of the derivative of the function

$f(x)$ when $x = 3$)

There is no work for this.

$slope\ of\ tangent\ line = value\ of\ derivaitve$

Answer: $\boldsymbol{f'(3) = 3}$ (The first 3 is the x-coordinate of the point $(3,0)$
The second 3 is the slope of the tangent line.)



(5, 6)
(3, 0)

This fact will be important and save us a lot of time if we use the fact instead of creating it each time

$$\boldsymbol{(x+h)^2 = x^2 + 2xh + h^2}$$

This is the algebra used to create this formula

$$(x+h)^2 = (x+h)(x+h) = xx + xh + hx + hh = x^2 + 1xh + 1xh + h^2 = x^2 + 2xh + h^2$$

Example - Definition of derivative:

Given: $f(x) = 3x^2 - 5x + 4$

a) Use the definition of the derivative to find f'(x)
b) Find f'(4)

a) The formula needed for part a: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f(x) = 3x^2 - 5x + 4$

Create 2 points

Use the given x as the x-coordinate of the first point

First point: $(x, \ \ 3x^2 - 5x + 4)$

Use $x + h$ as the x-coordinate of the second point.

Find the y-coordinate of the second point:

$f(x + h) = 3(x + h)^2 - 5(x + h) + 4$

$= 3(x^2 + 2xh + h^2) - 5(x + h) + 4$

$= 3x^2 + 6xh + 3h^2 - 5x - 5h + 4$

Second point: $(x + h, \ 3x^2 + 6xh + 3h^2 - 5x - 5h + 4)$

$Instantaneous\ rate\ of\ change =$

$\lim_{h \to 0} \frac{(3x^2+6xh+3h^2-5x-5h+4)-(3x^2-5x+4)}{h}$

$= \lim_{h \to 0} \frac{3x^2+6xh+3h^2-5x-5h+4-3x^2+5x-4}{h}$

$= \lim_{h \to 0} \frac{6xh+3h^2-5h}{h} = \lim_{h \to 0} \frac{h(6x+3h-5)}{h} = \lim_{h \to 0}(6x + 3h - 5) = 6x + 3(0) - 5 = 6x - 5$

Answer: $f'(x) = 6x - 5$

b) $f'(4) = 6(4) - 5 = 24 - 5 = 19$

Answer: $f'(4) = 19$

Example - Equation of a tangent line:

Given: $f(x) = 3x^2 - 5x + 4; \quad x = -1$

a) Find a formula to find the slope of a tangent line. (Note a derivative is a formula that can be used to find the slope of a tangent line.)

We found the derivative of this function in the last example.

$\underline{\boldsymbol{Answer}\text{: } \boldsymbol{f'(x) = 6x - 5}}$

b) Find the equation of the tangent line when $x = -1$.

We need a point and a slope to find an equation of a line.

Slope is found by substituting $x = -1$ into the derivative.

$slope = m = f'(-1) = 6(-1) - 5 = -6 - 5 = -11$
$m = -11$

The x-coordinate of the point we need is $x = -1$.

The y-coordinate is found by substituting $x = -1$ into the original function.

$y = f(-1) = 3(-1)^2 - 5(-1) + 4 = 12$

$point = (-1,12)$

Now use $y - y_1 = m(x - x_1)$

$with\ m = -11,\ x_1 = -1,\ y_1 = 12\ to\ find\ the\ desired\ equation.$

$y - 12 = -11(x - (-1))$

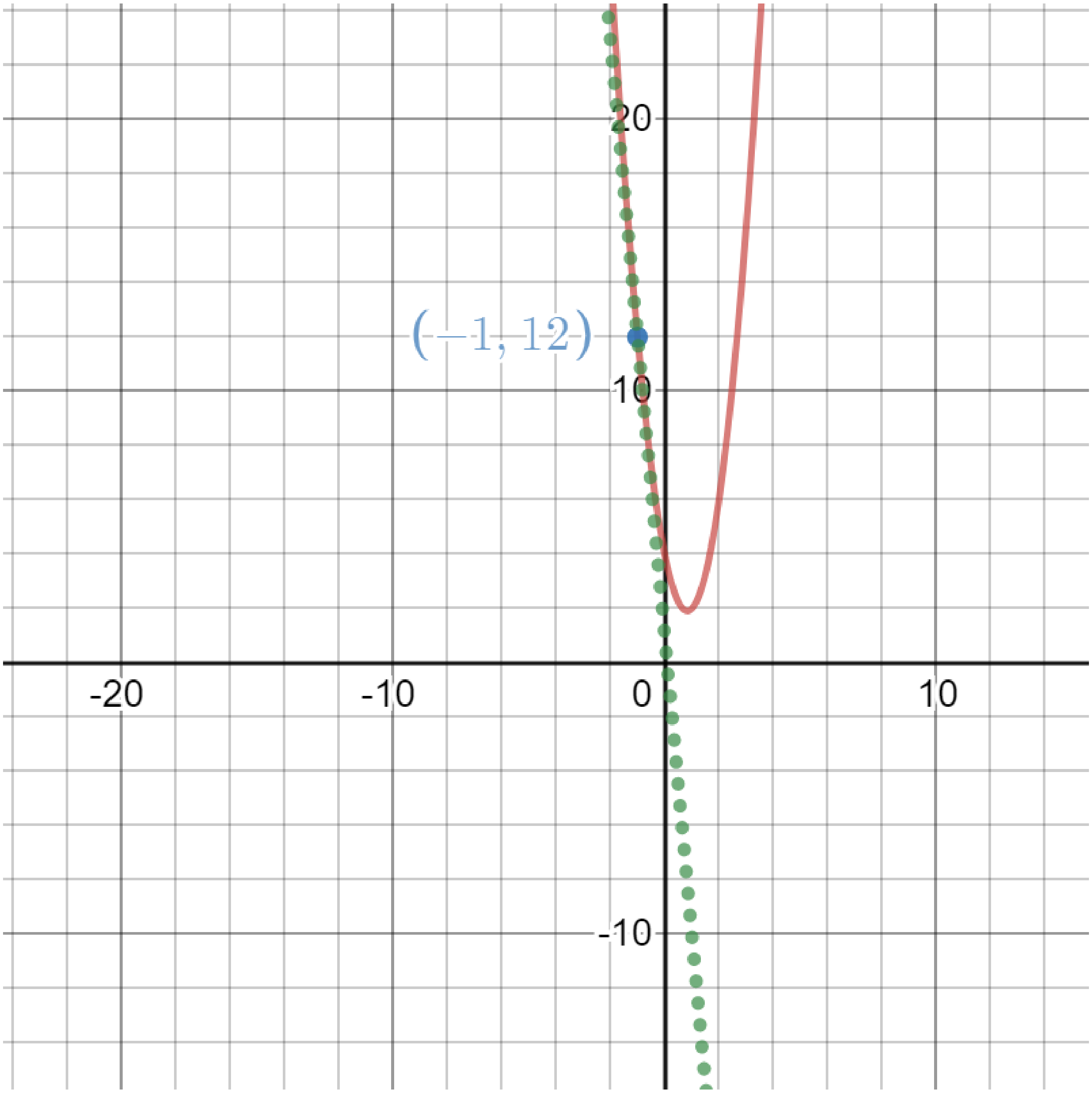
$y - 12 = -11(x + 1)$

$y - 12 = -11x - 11$

Answer: $\underline{\boldsymbol{y = -11x + 1}}$

Here is a graph of $f(x) = 3x^2 - 5x + 4$, and the line $y = -11x + 1$

You can kind of see the line is tangent to the parabola at the point $(-1,12)$



Marginal Profit

- **Marginal profit** is the profit earned by a firm when one additional unit (or marginal unit) is produced and sold.
- <u>The formula to compute marginal profit is the derivative of a profit formula.</u>
- Marginal profit is the difference between marginal revenue and marginal cost.
- Marginal profit analysis is helpful because it can help determine whether to expand or contract production or to stop production altogether, a moment known as the shutdown point.
- Under mainstream economic theory, a company will maximize its overall profits when marginal cost equals marginal revenue, or when marginal profit is exactly zero.

Marginal revenue is the additional revenue earned when one additional unit is sold.

Marginal cost is the additional cost generated when one additional unit is produced.

Understanding Marginal Profit

Marginal profit is different from average profit, net profit, and other measures of profitability in that it looks at the money to be made on producing one additional unit. It accounts for scale of production because as a firm gets larger, its cost structure changes and, depending on economies of scale, profitability can either increase or decrease as production ramps up.

Economies of scale refer to the situation where marginal profit increases as the scale of production is increased. At a certain point, the marginal profit will become zero and then turn negative as scale increases beyond its intended capacity. At this point, the firm experiences diseconomies of scale.

Companies will thus tend to increase production until marginal cost equals marginal product, which is when marginal profit equals zero. In other words, when marginal cost and marginal product (revenue) is zero, there is no additional profit earned for producing an added unit.

If the marginal profit of a firm turns negative, its management may decide to scale back production, halt production temporarily, or abandon the business altogether if it appears that positive marginal profits will not return.

Example:

A real estate has a 100-unit apartment and plans to rent out the apartments. The monthly profit generated by renting out x units of the apartment is given by

$$P(x) = -10x^2 + 1760x - 50{,}000$$

a) Find the marginal profit function P'(x)
b) Find P'(50) and interpret your answer

a) The marginal profit function is derivative of the profit function.

Create 2 points

Use the given x as the x-coordinate of the first point

First point: $(x,\ -10x^2 + 1760x - 50{,}000)$

Use $x + h$ as the x-coordinate of the second point.

Find the y-coordinate of the second point:

$$f(x+h) = -10(x+h)^2 + 1760(x+h) - 50000$$
$$= -10(x^2 + 2xh + h^2) + 1760(x+h) - 50000$$
$$= -10x^2 - 20xh - 30h^2 + 1760x + 1760h - 50000$$

Second point:

$$(x+h, -10x^2 - 20xh - 30h^2 + 1760x + 1760h - 50000)$$

$$Marginal\ profit\ P'(x) =$$
$$\lim_{h\to 0} \frac{(-10x^2 - 20xh - 30h^2 + 1760x + 1760h - 50000) - (-10x^2 + 1760x - 50000)}{h}$$

$$= \lim_{h\to 0} \frac{-10x^2 - 20xh - 30h^2 + 1760x + 1760h - 50000 + 10x^2 - 1760x + 50000}{h}$$

$$= \lim_{h\to 0} \frac{-20xh - 30h^2 + 1760h}{h}$$

$$= \lim_{h\to 0} \frac{h(-20x - 30h + 1760)}{h} = \lim_{h\to 0}(-20x - 30h + 1760) = -20x - 30(0) + 1760$$

Answer: $P'(x) = -20x + 1760$

b) $P'(50) = -20(50) + 1760 = -1000 + 1760 = 760$

Answer:

$\underline{P'(50) = 760}$

The company will earn an additional \$760 of profit when the next (51^{st}) unit is produced and sold.

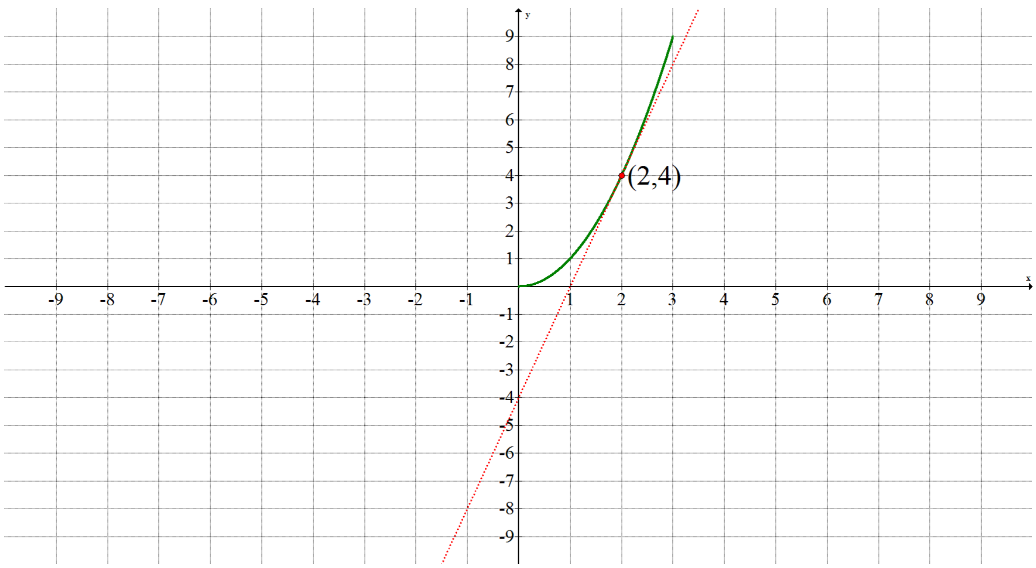
(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

#1-4: The graph of $f(x)$ is given below.

a) Find the slope of the tangent line at the given point (x,y)

b) Find $f'(x)$ at the given point (x, y)

1)



2)

#1-4: The graph of $f(x)$ is given below.

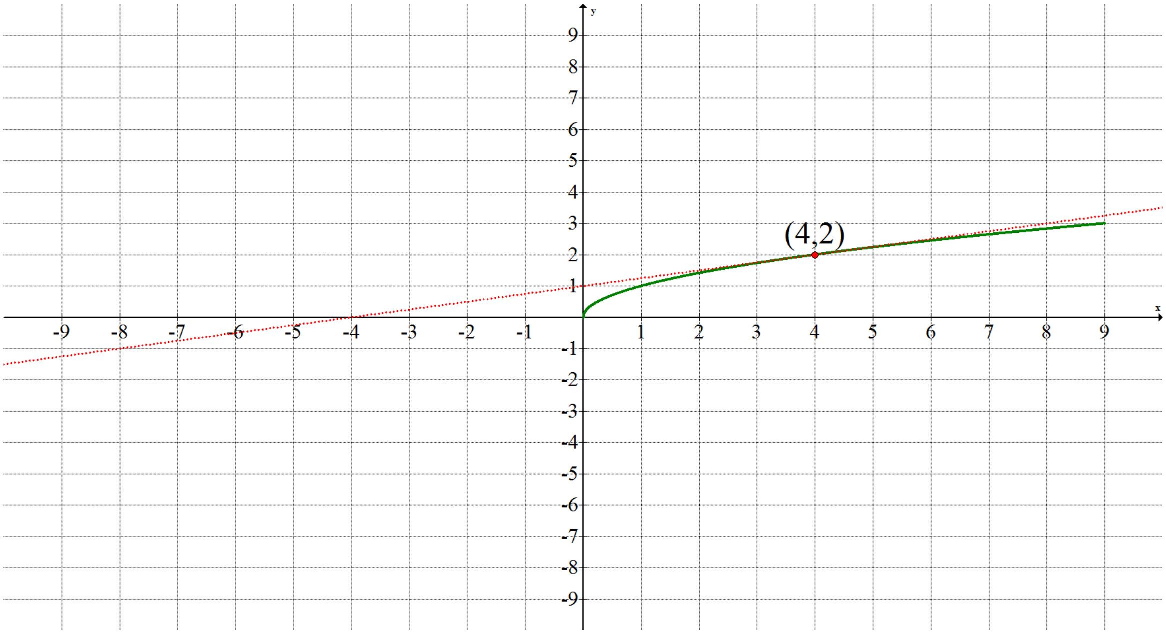
a) Find the slope of the tangent line at the given point (x,y)

b) Find $f'(x)$ at the given point (x, y)

Answers: 2a) $m = -4$ b) $f'(-2) = -4$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

3)

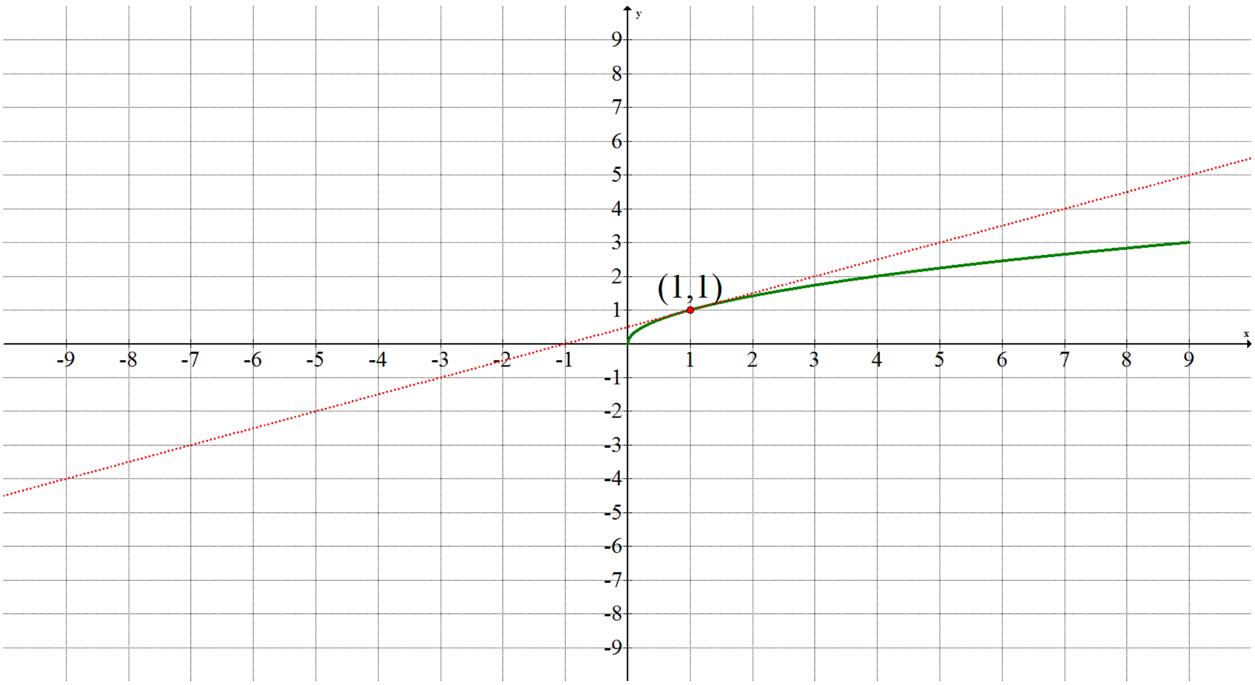


4)

#1-4: The graph of $f(x)$ is given below.

a) Find the slope of the tangent line at the given point (x,y)

b) Find $f'(x)$ at the given point (x, y)



Answers: 4a) $m = \frac{1}{2}$ 4b) $f'(1) = \frac{1}{2}$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

#5-14: For each problem complete the following.

a) Use the definition of the derivative to find $f'(x)$

b) Find $f'(4)$

5) $f(x) = x^2 + 3x - 4$

6) $f(x) = x^2 - 5x + 7$

a) Use the definition of the derivative to find f’(x)

b) Find f’(4)

$answers$: $a)$ $f'(x) = 2x - 5$

b) $f'(4) = 3$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

7) $f(x) = 6x^2 + 12$

8) $f(x) = 3x^2 - 4$

a) Use the definition of the derivative to find f'(x)

b) Find f'(4)

Answers: a) $f'(x) = 6x$

b) $f'(4) = 24$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

9) $f(x) = 3x^2 - 4x + 2$

10) $f(x) = 5x^2 - 6x + 1$

a) Use the definition of the derivative to find $f'(x)$

b) Find $f'(4)$

Answers: a) $f'(x) = 10x - 6$

b) $f'(4) = 34$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

11) f(x) = $\frac{2}{x}$

12) f(x) = $\frac{3}{x}$

a) Use the definition of the derivative to find f'(x)

b) Find f'(4)

Answers: a) $f'(x) = -\frac{3}{x^2}$

b) $m = f'(4) = -\frac{3}{16}$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

13) f(x) = $\frac{5}{x}$

14) f(x) = $\frac{7}{x}$

a) Use the definition of the derivative to find f'(x)

b) Find f'(4)

Answers: a) $f'(x) = -\frac{7}{x^2}$

b) $f'(4) = -\frac{7}{16}$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

#15-24: For each problem complete the following:

a) Find a formula to find the slope of a tangent line.

b) Find the equation of the tangent line through the given value of x.

15) $f(x) = x^2 + x - 4$, $x = 3$

16) $f(x) = x^2 - 2x + 3$, $x = 4$

a) Find a formula to find the slope of a tangent line.

b) Find the equation of the tangent line through the given value of x.

Answers: $a)$ $f'(x) = 2x - 2$

b) $y = 6x - 13$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

17) f(x) = $3x^2 + 7$, x = 5

18) f(x) = $2x^2 - 1$, x = -2

a) Find a formula to find the slope of a tangent line.

b) Find the equation of the tangent line through the given value of x.

Answers: $a)\ f'(x) = 4x$

b) $y = -8x - 9$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

19) $f(x) = 3x^2 - 2x + 3$, $x = 1$

20) $f(x) = 5x^2 - 2x + 8$, $x = 0$

a) Find a formula to find the slope of a tangent line.

b) Find the equation of the tangent line through the given value of x.

Answers: $a)\ f'(x) = 10x - 2$

b) $y = -2x + 8$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

21) f(x) = $\frac{-8}{x}$, x = -3

22) f(x) = $-\frac{6}{x}$, x = -5

a) Find a formula to find the slope of a tangent line.

b) Find the equation of the tangent line through the given value of x.

$answers$: $a)$ $f'(x) = \frac{6}{x^2}$

b) $y = \frac{1}{5}x + \frac{11}{5}$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

23) f(x) = $\frac{-3}{x}$, x = 2

24) f(x) = $\frac{-4}{x}$, x = 2

a) Find a formula to find the slope of a tangent line.

b) Find the equation of the tangent line through the given value of x.

Answers: $a)\ f'(x) = \frac{4}{x^2}$

b) $y = 2x - 4$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

25) A toy rocket is launched straight up so that its height *s*, in meters, at time *t*, in seconds, is given by $s(t)=-2t^2+30t+5$.

a) Find s'(t)

b) Find s'(2) and interpret your answer

26) If a baseball is projected upward from ground level with an initial velocity of 64 feet per second, then its height is a function of time, given by $s(t) = -16t^2 + 64t$

a) Find $s'(t)$

b) Find $s'(2)$ and interpret your answer

Answer a) $s'(t) = -32t + 64$

b) $s'(2) = 0$ objects velocity is 0 feet per second. (this is the moment where it stops rising and begins falling)

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

27) A pebble is dropped from a cliff, 50 m high. After *t* sec, the pebble is *s* meters above the ground, where $s(t)=50-2t^2$.

a) Find s'(t)

b) Find s'(1) and interpret your answer

28) A cannon ball is dropped from a building. Suppose that the height of the cannon ball (in meters) after t seconds is given by the quadratic function:

$f(t) = -4.4t^2 + 50$.

a) Find f'(t)

b) Find f'(1) and interpret your answer

Answers: a) $answer\ f'(t) = -8.8t$

b) $answer\ f'(a) = \ -8.8$

cannon ball's velocity is -8.8 meters per second at 1 second.

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

29) The profit from sale of x car seats for is given by the formula:

$P(x)= 45x - 0.0025x^2 - 5000$

a) Find the marginal profit function $P'(x)$

b) Find $P'(800)$ and interpret your answer

30) The profit from sale of x cell phones is given by the formula:

$P(x)= 450x - 0.055x^2 - 300000$

a) Find the marginal profit function $P'(x)$

b) Find $P'(1000)$ and interpret your answer

Answers a) $P'(x) = 450 - 0.11x$

b) $P'(1000) = 340$

$an\ additional\ \$340\ of\ profit\ will\ be\ earned\ when\ the\ 1001^{st} cell\ phone\ is\ produced\ and\ sold)$

(Minimum Homework: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21, 25, 29)

31) The cost of manufacturing x chairs is given by the function:

$C(x) = x^2 + 40x + 800$

a) Find the marginal cost function $C'(x)$

b) Find $C'(30)$ and interpret your answer

32) The cost of manufacturing x books is given by the function:

$C(x) = x^2 + 30x + 50$

a) Find the marginal cost function $C'(x)$

b) Find $C'(20)$ and interpret your answer